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2026 Volume 4, Issue 3
Educational Innovation Research
ISSN: 3029-1852 (Online)
ISSN: 3029-1844 (Print)

Identities Involving Hyperbolic Functions and Chebyshev Polynomials

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Abstract: This paper discusses the power-sum problems of hyperbolic since and cosine functions by using the fundamental properties of Chebyshev polynomials of the first and second kinds. Based on their recurrence relations, explicit forms and related functional identities, we derive a set of combinatorial summation formulas for power sums of sinh(x) and cosh(x). The theorems and corollaries presented in this work further extend the known results concerning special polynomials and hyperbolic functions. These identities are applicable to number theory and combinatorial analysis, and enrich the practical applications of Chebyshev polynomials in summation problems.

Keywords: Hyperbolic functions; Chebyshev polynomials; Power index sums

Online publication: March 26, 2026

1. Introduction

The Chebyshev polynomials of the first kind {Tn(x)} and the second kind {Un(x)} are defined respectively as

$$Tn+2(x) = 2xTn+1(x) — Tn(x) \tag{1.1}$$

and

$$Un+2(x) = 2xUn+1(x) — Un(x), \tag{1.2}$$

where $n \geq 0$, T0(x) = 1, T1(x) = x, U0(x) = 1 and U1(x) = 2x. The explicit expressions of {Tn(x)} and {Un(x)} are as follows

$$\begin{aligned} T_n(x) &= \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right] \\ &= \frac{n}{2}\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k} \end{aligned} \tag{1.3}$$

And

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right]$$
$$= \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \binom{n-k}{k}(2x)^{n-2k}. \quad (1.4)$$

It is well-known that

Tn(cosh(x)) = cosh(nx) and Un $(\cosh(x)) = \frac{\sinh((n+1)x)}{\sinh(x)}$, (1.5)

where cosh $(x) = \frac{e^x+e^{-x}}{2}$ and sinh $(x) = \frac{e^x-e^{-x}}{2}$ are the hyperbolic functions.

Many papers have studied the properties of the Chebyshev polynomials {Tn(x)} and {Un(x)} (see for details [1–10]). For example, Lv and Shen studied the power sum problems for the sin x and cos x functions and obtained some interesting computational formulas by using the properties of the Chebyshev polynomials.

In this paper we shall use the properties of the Chebyshev polynomials to study the power mean of hyperbolic functions by developing the ideas of. Our results are as following.

Theorem **1.1.** For any non-negative integer p and positive integers q and n, then we have

$$\sum_{a=0}^{q}\cosh^n\left(\frac{ap}{q}\right) = \frac{1}{2} + \frac{1}{2\cdot 2^n}\sum_{k=0}^{n}\binom{n}{k}U_{2q}\left(\cosh\left(\frac{(n-2k)p}{2q}\right)\right).$$

Theorem 1.2. For any non-negative integer p and positive integers q and n, then we have

$$\sum_{a=0}^{q}\sinh^{2n}\left(\frac{ap}{q}\right) = \frac{1}{2\cdot 4^n}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right).$$

The following corollary can be obtained from Theorems 1.1, 1.2 and formula (1.4).

Corollary 1.3. For any positive integers q and n with 2q ≤ n, p be any non-negative integer, we have

$$\sum_{a=0}^{q}T_{2a}^{n}\left(\cosh\left(\frac{p}{2q}\right)\right) = \frac{1}{2} + \frac{1}{2}\sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}T_{n-2j-2l}^{n}\left(\cosh\left(\frac{p}{2q}\right)\right),$$

and

$$\sum_{a=1}^{q}U_{2a-1}^{2n}\left(\cosh\left(\frac{p}{2q}\right)\right) = \frac{1}{2}\sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}U_{n-2j-2l-1}^{2n}\left(\cosh\left(\frac{p}{2q}\right)\right).$$

Taking $p = 0$ and n = 2q in Corollary 1.3 we immediately deduce the following corollary.

Corollary 1.4. For any positive integer q, we have

$$\sum_{j=0}^{q}\sum_{l=0}^{2q-2j}(-1)^j\binom{2q-j}{j,l} = 2q+1$$

and

$$\sum_{j=0}^{q}\sum_{l=0}^{2q-2j}(-1)^j\binom{2q-j}{j,l}(q-j-l)^{4q} = 2\sum_{a=1}^{q}a^{4q},$$

where (2 \s \up q 11(q) ,l j) = (2qj—j) · (2q \s \up 11(—) 2j).

2. Preliminaries on Chebyshev polynomials

To complete the proofs of our results we need some properties of Chebyshev polynomials, which we summarize as the following lemmas.

Lemma 2.1. For any non-negative integer n, we have

$$U_n(i\sinh(x)) = \begin{cases} \frac{(-1)^k\sinh(2kx)}{i\cosh(x)}, & if\ \ n=2k-1, \\ \frac{(-1)^k\cosh((2k+1)x)}{\cosh(x)}, & if\ \ n=2k. \end{cases}$$

Proof. Applying (1.4), we have

$$\begin{aligned} U_n(i\sinh(x)) &= \frac{1}{2\sqrt{(i\sinh(x))^2-1}}\left[\left(i\sinh(x)+\sqrt{(i\sinh(x))^2-1}\right)^{n+1}\right. \\ &\quad \left.-\left(i\sinh(x)-\sqrt{(i\sinh(x))^2-1}\right)^{n+1}\right] \\ &= \frac{1}{2i\cosh(x)}\left[(i\sinh(x)+i\cosh(x))^{n+1}-(i\sinh(x)-i\cosh(x))^{n+1}\right] \\ &= \frac{1}{2i\cosh(x)}\left[(ie^{x})^{n+1}-\left(-ie^{-x}\right)^{n+1}\right] \\ &= \begin{cases} \frac{(-1)^k\sinh(2kx)}{i\cosh(x)}, & \text{if}\ \ n=2k-1, \\ \frac{(-1)^k\cosh((2k+1)x)}{\cosh(x)}, & \text{if}\ \ n=2k. \end{cases} \end{aligned}$$

This proves **Lemma 2.1**

Lemma 2.2. For any non-negative integer n, we have the identities

$$x^{2n} = \frac{\binom{2n}{n}}{4^n}T_0(x) + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}T_{2k}(x) \tag{2.1}$$

and

$$x^{2n+1} = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{n-k}T_{2k+1}(x). \tag{2.2}$$

Proof. This is **Lemma 4** [11]. □

Lemma 2.3. For any real numbers α and β, we have

$$\sinh(\alpha)+\sinh(\beta) = 2\sinh\left(\frac{\alpha+\beta}{2}\right)\cosh\left(\frac{\alpha-\beta}{2}\right).$$

Lemma 2.4. For any positive integer $n \geq 2$, we have

$$T_n(x) = \frac{1}{2}U_n(x) - \frac{1}{2}U_{n-2}(x), \tag{2.3}$$

and further,

$$\sum_{k=0}^{n}T_k(x) = \frac{1}{2}U_n(x) + \frac{1}{2}U_{n-1}(x) + \frac{1}{2}. \tag{2.4}$$

3. Proof of theorems

Now we prove our results.

3.1. Proof of theorem 1.1

Replace cosh(父) of (2.1) in Lemma 2.2 with 父 , we have

$$\cosh^{2n}(x) = \frac{\binom{2n}{n}}{4^n} T_0(\cosh(x)) + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k} T_{2k}(\cosh(x)).$$

Then from the first formula of (1.5), Lemma 2.1, Lemma 2.3 and (2.4) of Lemma 2.4, we have q

$$\begin{aligned}
\sum_{a=0} \cosh 2n \\
\left(\frac{ap}{q}\right) &= \frac{\binom{2n}{n}}{4^n}\sum_{a=0}^{q} T_0\left(\cosh\left(\frac{ap}{q}\right)\right) + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\sum_{a=0}^{q} T_{2k}\left(\cosh\left(\frac{ap}{q}\right)\right) \\
&= \frac{\binom{2n}{n}}{4^n}\sum_{a=0}^{q} 1 + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\sum_{a=0}^{q} T_a\left(\cosh\left(\frac{2kp}{q}\right)\right) \\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k} \\
&\quad \times \left(U_q\left(\cosh\left(\frac{2kp}{q}\right)\right) + U_{q-1}\left(\cosh\left(\frac{2kp}{q}\right)\right) + 1\right)
\end{aligned}$$

Further,

$$\begin{aligned}
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\left(\frac{\sinh\left((q+1)\frac{2kp}{q}\right) + \sinh\left(q\frac{2kp}{q}\right)}{\sinh\left(\frac{2kp}{q}\right)} + 1\right) \\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\left(\frac{2\sinh\left((2q+1)\frac{kp}{q}\right)\cosh\left(\frac{kp}{q}\right)}{2\sinh\left(\frac{kp}{q}\right)\cosh\left(\frac{kp}{q}\right)} + 1\right) \\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\left(\frac{\sinh\left((2q+1)\frac{kp}{q}\right)}{\sinh\left(\frac{kp}{q}\right)} + 1\right) \\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\left(U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right) + 1\right).
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{a=0}^{q}\cosh^{2n}\left(\frac{ap}{q}\right) &= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\left(U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right) + 1\right) \\
&= (2q+1))\frac{\binom{2n}{n}}{2\cdot 4^n} + \frac{\binom{2n}{n}}{2\cdot 4^n} + \frac{1}{2\cdot 4^n}\left(\sum_{k=1}^{n}\binom{2n}{n-k}1 + \sum_{k=1}^{n}\binom{2n}{n+k}1\right. \\
&\quad \left. + \sum_{k=1}^{n}\binom{2n}{n-k}U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right) + \sum_{k=1}^{n}\binom{2n}{n+k}U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right)\right) \\
&= \frac{1}{2\cdot 4^n}\left(\sum_{k=0}^{2n}\binom{2n}{k}1 + \sum_{k=0}^{n}\binom{2n}{n-k}U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{k=1}^{n} \binom{2n}{n+k} U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right)\Bigg) \\
&= \frac{1}{2} + \frac{1}{2\cdot 4^n}\left(\sum_{k=0}^{n} \binom{2n}{n-k} U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right)\right. \\
&\quad + \sum_{k=1}^{n} \binom{2n}{n+k} U_{2q}\left(\cosh\left(\frac{kp}{q}\right)\right)\Bigg) \\
&= \frac{1}{2} + \frac{1}{2\cdot 4^n}\left(\sum_{k=0}^{n} \binom{2n}{k} U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right)\right. \\
&\quad + \sum_{k=n+1}^{2n} \binom{2n}{k} U_{2q}\left(\cosh\left(\frac{|n-k|p}{q}\right)\right)\Bigg) \\
&= \frac{1}{2} + \frac{1}{2\cdot 4^n}\left(\sum_{k=0}^{n} \binom{2n}{k} U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right)\right. \\
&\quad + \sum_{k=n+1}^{2n} \binom{2n}{k} U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right)\Bigg) \\
&= \frac{1}{2} + \frac{1}{2\cdot 4^n}\sum_{k=0}^{2n} \binom{2n}{k} U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right).
\end{aligned}
$$

p

Replace $\cosh\left(\frac{a}{q}\right)$ of (2.2) in Lemma 2.2 with x, and sum a from 0 to q, we have

$$
\begin{aligned}
\sum_{a=0}^{q} \cosh^{2n+1}\left(\frac{ap}{q}\right) &= \frac{1}{4^n}\sum_{k=0}^{n} \binom{2n+1}{n-k} \sum_{a=0}^{q} T_{2k+1}\left(\cosh\left(\frac{ap}{q}\right)\right) \\
&= \frac{1}{4^n}\sum_{k=0}^{n} \binom{2n+1}{n-k} \sum_{a=0}^{q} T_{a}\left(\cosh\left(\frac{(2k+1)p}{q}\right)\right).
\end{aligned}
$$

In a similar way we have

$$
\sum_{a=0}^{q} \cosh^{2n+1}\left(\frac{ap}{q}\right) = \frac{1}{2} + \frac{1}{4^{n+1}}\sum_{k=0}^{2n+1} \binom{2n+1}{k} U_{2q}\left(\cosh\left(\frac{((2n+1)-2k)p}{2q}\right)\right).
$$

So we get

$$
\sum_{a=0}^{q} \cosh^{n}\left(\frac{ap}{q}\right) = \frac{1}{2} + \frac{1}{2\cdot 2^{n}}\sum_{k=0}^{n} \binom{n}{k} U_{2q}\left(\cosh\left(\frac{(n-2k)p}{2q}\right)\right).
$$

This proves Theorem 1.1

3.2. Proof of theorem 1.2

Replace cosh(x) of (2.1) in Lemma 2.2 with x,we have

$$
(i\sinh(x))^{2n} = (-1)^n \sinh^{2n}(x) = \frac{\binom{2n}{n}}{4^n} T_0(i\sinh(x)) + \frac{2}{4^n}\sum_{k=1}^{n} \binom{2n}{n-k} T_{2k}(i\sinh(x)).
$$

Then from the second formula of (1.5), Lemma 2.1, Lemma 2.3 and (2.4) of Lemma 2.4, we have

$$
\begin{aligned}
&\sum_{a=0}^{q}(-1)^n \sinh^{2n}\left(\frac{ap}{q}\right)\\
&= \frac{\binom{2n}{n}}{4^n}\sum_{a=0}^{q} T_0\left(i\sinh\left(\frac{ap}{q}\right)\right) + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\sum_{a=0}^{q} T_{2k}\left(i\sinh\left(\frac{ap}{q}\right)\right)\\
&= \frac{\binom{2n}{n}}{4^n}\sum_{a=0}^{q} 1 + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}\sum_{a=0}^{q} T_{2k}\left(i\sinh\left(\frac{ap}{q}\right)\right)\\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}(-1)^k\sum_{a=0}^{q}\cosh\left(\frac{2kap}{q}\right)\\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{2}{4^n}\sum_{k=1}^{n}\binom{2n}{n-k}(-1)^k\sum_{a=0}^{q}T_a\left(\cosh\left(\frac{2kp}{q}\right)\right)\\
&= (q+1)\frac{\binom{2n}{n}}{4^n} + \frac{1}{2\cdot 4^n}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^{n-k}U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right)\\
&= (-1)^n\frac{1}{2\cdot 4^n}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right).
\end{aligned}
$$

Furthermore, we get

$$
\sum_{a=0}^{q}\sinh^{2n}\left(\frac{ap}{q}\right) = \frac{1}{2\cdot 4^n}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k U_{2q}\left(\cosh\left(\frac{(n-k)p}{q}\right)\right).
$$

This proves Theorem 1.2

3.3. Proof of corollary 1.3

From formula (1.4) and Theorem 1.1, we get

$$
\begin{aligned}
U_{2q}\left(\cosh\left(\frac{(n-2k)p}{2q}\right)\right) &= \sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\left(2\left(\cosh\left(\frac{(n-2k)p}{2q}\right)\right)\right)^{n-2j}\\
&= \sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}e^{\frac{p(n-2k)(n-2j-2l)}{2q}}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}U_{2q}\left(\cosh\left(\frac{(n-2k)p}{2q}\right)\right)\\
&= \sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}e^{-\frac{p(n-2j-2l)}{2q}n}\sum_{k=0}^{n}\binom{n}{k}e^{\frac{p(n-k)(n-2j-2l)}{q}}\\
&= \sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}\left(e^{\frac{p(n-2j-2l)}{2q}} + e^{-\frac{p(n-2j-2l)}{2q}}\right)^n\\
&= 2^n\sum_{j=0}^{q}(-1)^j\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}\cosh^n\left(\frac{p(n-2j-2l)}{2q}\right).
\end{aligned}
$$

Furthermore, from the second formula of (1.5) we get

$$\sum_{a=0}^{q} T_{2a}^{n}\left(\cosh\left(\frac{p}{2q}\right)\right)=\sum_{a=0}^{q}\cosh^{n}\left(\frac{ap}{q}\right)$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{j=0}^{q}(-1)^{j}\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}\cosh^{n}\left(\frac{p(n-2j-2l)}{2q}\right)$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{j=0}^{q}(-1)^{j}\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}T_{n-2j-2l}^{n}\left(\cosh\left(\frac{p}{2q}\right)\right).$$

Noting that

$$\sum_{a=0}^{q}\sinh^{2n}\left(\frac{ap}{q}\right)=\sum_{a=1}^{q}\sinh^{2n}\left(\frac{ap}{q}\right),$$

thus similarly we have

$$\sum_{a=1}^{q} U_{2a-1}^{2n}\left(\cosh\left(\frac{p}{2q}\right)\right)=\frac{1}{2}\sum_{j=0}^{q}(-1)^{j}\binom{2q-j}{j}\sum_{l=0}^{n-2j}\binom{n-2j}{l}U_{n-2j-2l-1}^{2n}\left(\cosh\left(\frac{p}{2q}\right)\right).$$

This proves Corollary 1.3

4. Conclusion

This paper establishes several new summation identities for power sums of sinh(x) and cosh(x)with the help of Chebyshev polynomials. By converting hyperbolic summation problems into polynomial calculations through recurrence formulas and explicit expressions of Chebyshev polynomials, we obtain compact combinatorial expressions for the target power sums.

Our results broaden the scope of Chebyshev polynomials in hyperbolic function research and provide alternative tools for studying summation identities of special functions. In follow-up work, we will extend this method to other orthogonal polynomials and explore more identities involving trigonometric and hyperbolic sums.

Funding

National Natural Science Foundation of China (Project No.: 12071368); The Science and Technology Program of Shaanxi Province of China (Project No.: 2024JC-JCQN-04); Shaanxi Fundamental Science Research Project for Mathematics and Physics (Project No.: 22JSY017)

Disclosure statement

The author declares no conflict of interest.

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