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Research on A Class of Infinite Radical Nested Limit Problems Based on MATLAB

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Abstract: In this paper, we use the cosine function to calculate the limit of an infinite root nested, modify the constants in the topic, and use the same method to calculate the limit. We extend the topic to the general case, and derive a limit formula containing the anti cosine function. Then we use the analogical reasoning method to derive a limit formula containing the anti hyperbolic cosine function. The curve of the function is drawn with mobile MATLAB, and the correctness of the results is verified, it shows the powerful function of MATLAB.

Keywords: infinite radical nesting; analogical reasoning; visualization; MATLAB; limit

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1. Introduction

There is a limit problem of a sequence in the review textbook of advanced mathematics [1]

$$L=\lim_{n\to\infty}\left(\frac{2}{\sqrt{2}}\cdot\frac{2}{\sqrt{2+\sqrt{2}}}\cdot\ldots\cdot\frac{2}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}}\right) \tag{1}$$

This is a problem of the limit of a nested sequence with infinite roots, and many students do not know how to start. After searching on the Knowledge Network, it was found that someone has used geometric methods to find its limit. In fact, the calculation of this limit does not need to be so complicated, and the online solution has not been extended to general situations. Therefore, the solution in this article has good guiding significance for the teaching content of the limit part in higher mathematics.

The method for finding such limits has certain techniques. Firstly, this article uses cosine functions to obtain the correct results in detail. Secondly, the problem is deeply studied, and the constants are modified to calculate the correct results. Finally, the problem is further extended, and two limit formulas are derived based on different conditions, and the curves are plotted using a mobile version of MATLAB [3-7]

2. Answer to two questions

Let $a_1=\sqrt{2}=2\cos\frac{\pi}{4}$, Using the half angle formula: $1+\cos x=2\cos^2\frac{x}{2}$, available

$$a_2=\sqrt{2+a_1}=\sqrt{2+2\cos\frac{\pi}{4}}=2\cos\frac{\pi}{8}\cdot$$

Similarly, it can be concluded that

$$a_3=\sqrt{2+a_2}=\sqrt{2+2\cos\frac{\pi}{8}}=2\cos\frac{\pi}{16}\cdot$$

and so forth

$$a_n=\sqrt{2+a_{n-1}}=2\cos\frac{\pi}{2^{n+1}}\ .$$

Therefore, the limit is

$$L=\lim_{n\to\infty}(\frac{2}{a_1}\cdot\frac{2}{a_2}\cdot\frac{2}{a_3}\cdot\ldots\cdot\frac{2}{a_n})=\lim_{n\to\infty}\frac{2^n}{2^n\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cdot\ldots\cdot\cos\frac{\pi}{2^{n+1}}}$$

$$=\lim_{n\to\infty}\frac{\sin\frac{\pi}{2}}{\prod_{i=1}^{n}\cos\frac{\pi}{2^{i+1}}}=\lim_{n\to\infty}\frac{2\cos\frac{\pi}{4}\sin\frac{\pi}{4}}{\prod_{i=1}^{n}\cos\frac{\pi}{2^{i+1}}}$$

$$=\lim_{n\to\infty}\frac{2^2\cos\frac{\pi}{4}\cos\frac{\pi}{8}\sin\frac{\pi}{8}}{\prod_{i=1}^{n}\cos\frac{\pi}{2^{i+1}}}$$

$$=\lim_{n\to\infty}\frac{2^3\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\sin\frac{\pi}{16}}{\prod_{i=1}^{n}\cos\frac{\pi}{2^{i+1}}}$$

……

$$=\lim_{n\to\infty}\frac{2^n\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cdot\ldots\cdot\cos\frac{\pi}{2^{n+1}}\sin\frac{\pi}{2^{n+1}}}{\prod_{i=1}^{n}\cos\frac{\pi}{2^{i+1}}}$$

Simplify to obtain

$$L=\lim_{n\to\infty}2^n\sin\frac{\pi}{2^{n+1}}=\lim_{n\to\infty}\frac{2^n\pi}{2^{n+1}}=\frac{\pi}{2}\approx 1.5708 \qquad (2)$$

3. The limit of the sequence after constant modification

In equation (1), The denominator of the first term in equation (1) is $a_1=\sqrt{2}$, If we change the square root 2 to 3, it becomes a problem of finding the limit of the following sequence

$$L=\lim_{n\to\infty}(\underbrace{\frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{2+\sqrt{3}}}\cdot\ldots}_{n}\cdot\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{3}}}}}) \qquad (3)$$

Can be set $a_1=\sqrt{3}=2\cos\frac{\pi}{6}$, therefore

$$a_2=\sqrt{2+a_1}=\sqrt{2+2\cos\frac{\pi}{6}}=2\cos\frac{\pi}{12},$$

Similarly, it can be concluded that

$$a_3=\sqrt{2+a_2}=\sqrt{2+2\cos\frac{\pi}{12}}=2\cos\frac{\pi}{24},$$

and so forth

$$a_n=\sqrt{2+a_{n-1}}=2\cos\frac{\pi}{3\times 2^n}.$$

Its limit is $$L=\lim_{n\to\infty}(\frac{2}{a_1}\cdot\frac{2}{a_2}\cdot\frac{2}{a_3}\cdot\ldots\cdot\frac{2}{a_n})$$

$$=\lim_{n\to\infty}\frac{1}{\cos\frac{\pi}{6}\prod_{i=2}^{n}\cos\frac{\pi}{3\times 2^i}}.$$

Due to

$$1=2\sin\frac{\pi}{6}=2^2\cos\frac{\pi}{12}\sin\frac{\pi}{12}$$

$$=2^3\cos\frac{\pi}{12}\cos\frac{\pi}{24}\sin\frac{\pi}{24}$$

$$\ldots\ldots$$

$$=2^n\cos\frac{\pi}{12}\cdot\ldots\cdot\cos\frac{\pi}{3\times 2^n}\sin\frac{\pi}{3\times 2^n},$$

So

$$L=\lim_{n\to\infty}\frac{2^n\sin\frac{\pi}{3\times 2^n}}{\cos\frac{\pi}{6}}$$

$$=\lim_{n\to\infty}\frac{2^n\cdot\frac{\pi}{3\times 2^n}}{\frac{\sqrt{3}}{2}}=\frac{2\sqrt{3}\pi}{9}\approx 1.2092 \qquad (4)$$

If taken $a_1=\sqrt{4}$, obviously the limit $L=1$.

4. General formula for limit

If we change the square root 2 to the variable x, the problem becomes finding the following limit2, the problem becomes finding the following limits

$$f(x)=\lim_{n\to\infty}(\frac{2}{\sqrt{x}}\cdot\frac{2}{\sqrt{2+\sqrt{x}}}\cdot\ldots\cdot\frac{2}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{x}}}}}_{n}}) \qquad (5)$$

This limit is a function of x.

When $x=4$, it was evident that $f(x)=1$.

4.1. The limit formula represented by the inverse cosine function

When $0<x<4$, Can be set

$$a_1 = \sqrt{x} = 2\cos\frac{\alpha}{2}, \qquad (6)$$

Therefore

$$a_2 = \sqrt{2+a_1} = \sqrt{2+2\cos\frac{\alpha}{2}} = 2\cos\frac{\alpha}{4},$$

Similarly, it can be concluded that

$$a_3 = \sqrt{2+a_2} = \sqrt{2+2\cos\frac{\alpha}{4}} = 2\cos\frac{\alpha}{8},$$

and so forth $\qquad a_n = \sqrt{2+a_{n-1}} = 2\cos\frac{\alpha}{2^n}$.

Therefore, equation (5) can be rewritten as

$$g(x) = \lim_{n\to\infty}\left(\frac{2}{a_1}\cdot\frac{2}{a_2}\cdot\frac{2}{a_3}\cdot\ldots\cdot\frac{2}{a_n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{\prod_{i=1}^{n}\cos\frac{\alpha}{2^i}} = \lim_{n\to\infty}\frac{\sin\alpha}{\sin\alpha\prod_{i=1}^{n}\cos\frac{\alpha}{2^i}}.$$

Due to

$$\sin\alpha = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}$$

$$= 2^2\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\sin\frac{\alpha}{4}$$

$$\ldots\ldots$$

$$= 2^n\sin\frac{\alpha}{2^n}\prod_{i=1}^{n}\cos\frac{\alpha}{2^i},$$

so

$$g(x) = \lim_{n\to\infty}\frac{2^n\sin\frac{\alpha}{2^n}}{\sin\alpha} = \frac{\alpha}{\sin\alpha}. \qquad (7)$$

By using equation (6), we can obtain

$$\alpha = 2\arccos\frac{\sqrt{x}}{2},$$

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{1}{2}\sqrt{x(4-x)},$$

By substituting into equation (7), we can obtain

$$g(x) = \frac{4\arccos\frac{\sqrt{x}}{2}}{\sqrt{x(4-x)}}\left(0<x<4\right). \qquad (8)$$

This is the limit formula represented by the inverse cosine function

When $x=2$, available

$$g(2) = 2\arccos\frac{\sqrt{2}}{2} = \frac{\pi}{2},$$

This is the result of equation (2).

When $x=3$, available

$$g(3)=\frac{4\arccos(\sqrt{3}/2)}{\sqrt{3}}=\frac{2\sqrt{3}\pi}{9},$$

This is the result of equation (4).

In order to study the continuity of functions, let $t=\sqrt{x}/2$, standard $x=4t^2$. Equation (8) is converted to

$$g(t)=\frac{\arccos t}{t\sqrt{1-t^2}}(0<t<1).$$

When $x\to 4^-$, $t\to 1^-$. By using the Roberta Law, we can obtain

$$\lim_{x\to 4^-}g(x)=\lim_{t\to 1^-}g(t)=\lim_{t\to 1^-}\frac{1}{t}\lim_{t\to 1^-}\frac{\arccos t}{\sqrt{1-t^2}}$$

$$=\lim_{t\to 1^-}\frac{\frac{-1}{\sqrt{1-t^2}}}{\frac{-2t}{2\sqrt{1-t^2}}}=\lim_{t\to 1^-}\frac{1}{t}=1.$$

so: $\lim_{x\to 4^-}g(x)=1$, There exists a left limit, which is 1.

4.2. Limit formula represented by inverse hyperbolic cosine function

When $x>4$, for example $x=5$, Trigonometric functions are no longer applicable. Due to the similarity between trigonometric functions and hyperbolic functions, this article uses analogical reasoning to derive the case using $x>4$ hyperbolic functions. Identity equations are used:

$$1+\cosh x=2\cosh^2\frac{x}{2}, \tag{9}$$

Assume $$a_1=\sqrt{x}=2\cosh\frac{\alpha}{2}, \tag{10}$$

Available $$a_2=\sqrt{2+a_1}=\sqrt{2+2\cosh\frac{\alpha}{2}}=2\cosh\frac{\alpha}{4},$$

Similarly, it can be concluded that $a_3=\sqrt{2+a_2}=\sqrt{2+2\cosh\frac{\alpha}{4}}=2\cosh\frac{\alpha}{8}$,

and so forth $$a_n=\sqrt{2+a_{n-1}}=2\cosh\frac{\alpha}{2^n},$$

Therefore, equation (5) can be rewritten as

$$h(x)=\lim_{n\to\infty}(\frac{2}{a_1}\cdot\frac{2}{a_2}\cdot\frac{2}{a_3}\cdot\ldots\cdot\frac{2}{a_n})=\lim_{n\to\infty}\frac{1}{\prod_{i=1}^{n}\cosh\frac{\alpha}{2^i}}$$

$$=\lim_{n\to\infty}\frac{\sinh\alpha}{\sinh\alpha\prod_{i=1}^{n}\cosh\frac{\alpha}{2^i}}$$

$$=\lim_{n\to\infty}\frac{2^n\cosh\frac{\alpha}{2}\cdot\ldots\cdot\cosh\frac{\alpha}{2^n}\sinh\frac{\alpha}{2^n}}{\sinh\alpha\prod_{i=1}^{n}\cosh\frac{\alpha}{2^i}}$$

$$=\frac{1}{\sinh\alpha}\lim_{n\to\infty}2^n\sinh\frac{\alpha}{2^n}=\frac{\alpha}{\sinh\alpha}. \quad (11)$$

The identity of hyperbolic functions is

$$\cosh^2 x-\sinh^2 x=1. \quad (12)$$

By using equations (9), (10), and (12), equation (11) can be transformed into an expression for x

$$h(x)=\frac{4\operatorname{arccosh}\frac{\sqrt{x}}{2}}{\sqrt{x(x-4)}}(x>4). \quad (13)$$

This is the limit represented by the inverse hyperbolic cosine function. When $x=5$, It can be calculated using MATLAB $h(5)=0.8608$.

To study the continuity of functions, let $t=\sqrt{x}/2$, standard $x=4t^2$. Equation (13) is converted to

$$h(t)=\frac{\operatorname{arccosh} t}{t\sqrt{t^2-1}}(t>1).$$

The relationship between inverse hyperbolic cosine function and logarithmic composite function is

$$\operatorname{arccosh} t=\ln(t+\sqrt{t^2-1})(t>1). \quad (14)$$

Therefore, it can be concluded that

$$h(t)=\frac{\ln(t+\sqrt{t^2-1})}{t\sqrt{t^2-1}}(t>1). \quad (15)$$

When $x\to 4^+$, $t\to 1^+$. By using the Roberta Law, we can obtain

$$\lim_{x\to 4^+}h(x)=\lim_{t\to 1^+}h(t)=\lim_{t\to 1^+}\frac{1}{t}\lim_{t\to 1^+}\frac{\ln(t+\sqrt{t^2-1})}{\sqrt{t^2-1}}$$

$$=\lim_{t\to 1^+}\frac{\dfrac{1+2t/2\sqrt{t^2-1}}{t+\sqrt{t^2-1}}}{\dfrac{2t}{2\sqrt{t^2-1}}}=\lim_{t\to 1^+}\frac{1}{t}=1.$$

it is thus clear that: $\lim_{x\to 4^+}h(x)=1$, The function $h(x)$ has a right limit at $x=4$, which is 1.

In summary, the expression of equation (5) can be summarized as

$$f(x)=\begin{cases}\dfrac{4\arccos\frac{\sqrt{x}}{2}}{\sqrt{x(4-x)}} & ,0<x<4\\ 1 & ,x=4\\ \dfrac{4\arccos h\frac{\sqrt{x}}{2}}{\sqrt{x(x-4)}} & ,x>4\end{cases}$$

As can be seen from the above argumentation process, $\lim_{x\to 4^-}f(x)=f(4)=1$, $\lim_{x\to 4^+}f(x)=f(4)=1$, The function $f(x)$ is both left continuous and right continuous at the point $x=4$, therefore the function $f(x)$ is continuous.

5. Visualization of formulas

MATLAB is a high-level computer language with powerful computing and plotting functions[2]. With the mobile version of MATLAB, it is easy to calculate values and draw curves.

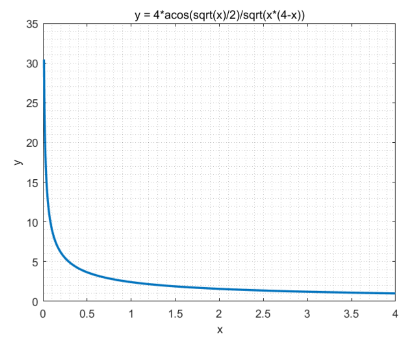

Figure 1. Inverse cosine function curve

From **Figure 1**, it can be seen that when x is equal to 2, $f(2)$ is approximately equal to 1.57; When x is equal to 3, $f(3)$is approximately equal to 1.21, which verifies the correctness of equation (8). Similarly, when x is equal to 4, $f(4)$ is always equal to 1.

Observing Figure 2, it is found that when x is equal to 5, $f(5)$ is approximately equal to 0.86, which verifies the correctness of equation (13). The verification of the above series of data shows the accuracy and reliability of the research method and results in this paper. **Figure 3** is perfectly connected by **Figure 1 and Figure 2**, which also verifies that the piecewise function $f(x)$ is continuous, and once again verifies the accuracy curve of the research results in this paper.

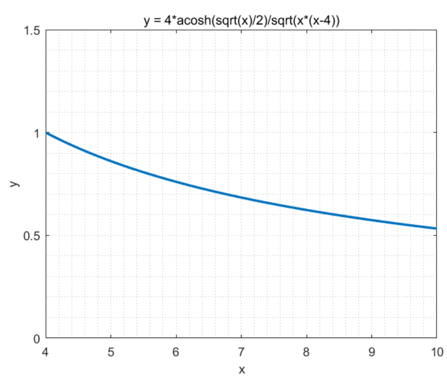

Figure 2. Inverse hyperbolic cosine function curve

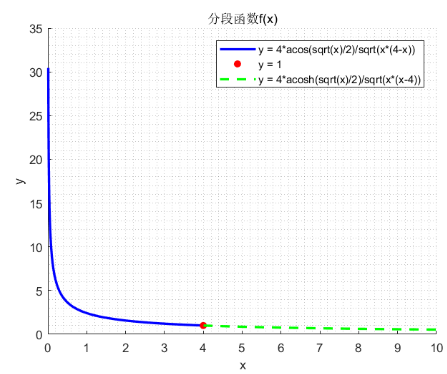

Figure 3. Segmented Function $f(x)$

MATLAB The function file in Figure 1 is as follows:

```
x = linspace(0.01,4,1000); %Define the scope of x
y = 4*acos(sqrt(x)/2)./sqrt(x.*(4-x)); %Calculate the value of function y
plot(x,y,’LineWidth’,2); %Draw a function graph,
set the line width to 2
grid minor; %Display fine grid lines
xlabel(‘x’); %Set x-axis label
ylabel(‘y’); %Set y-axis label
title(‘y = 4*acos(sqrt(x)/2)/sqrt(x*(4-x))’); %Set image title
```

The function file in Figure 2 is as follows:

```
X = linspace(4,10,1000); %Define the scope of x
Y = 4*acosh(sqrt(x3)/2)./sqrt(x3.*(x3-4)); %Calculate the value of function y
plot(x,y,’LineWidth’,2); %Draw a function graph, set the line width to 2
ylim([0,1.5]); %Set the range of the coordinate axis y
grid minor; %Display fine grid lines
```

```
xlabel('x'); %Set x-axis label
ylabel('y'); %Set y-axis label
title('y=4*acosh(sqrt(x)/2)/sqrt(x*(x-4))'); %Set image title
```

The function file in Figure 3 is as follows:

```
x1=linspace(0.01,3.99,1000);
x2=4;
x3=linspace(4,10,1000); %Define the scope of x
y1=4*acos(sqrt(x1)/2)./sqrt(x1.*(4-x1));
y2=1;
y3=4*acosh(sqrt(x3)/2)./sqrt(x3.*(x3-4)); %Calculate the value of y based on the given function
figure; %Create a new graphic window
hold on; %Draw multiple graphics in the same graphics window
plot(x1,y1,'b-','LineWidth',2); %Draw a blue solid line curve with a line width of 2
scatter(x2,y2,'r','filled'); %Draw a scatter plot filled in red
plot(x3,y3,'g--','LineWidth',2); %Draw a green dashed curve with a line width of 2
grid minor; %Display fine grid lines
hold off; %Stop adding graphics in the graphics window
legend('y=4*acos(sqrt(x)/2)/sqrt(x*(4-x))','y=1','y=4*acosh(sqrt(x)/2)/sqrt(x*(x-4))'); % add legend
xlabel('x'); %Add x-axis label
ylabel('y'); %Add y-axis label
title('piecewise functionf(x)'); %Add Title
```

6. Conclusion

The answers in this article have good guiding significance for the teaching content of extreme parts in higher mathematics. When learning higher mathematics, after solving each problem, one should think deeply, draw analogies, and deduce general formulas from special to general; Using analogical reasoning to study general formulas under different conditions and seek relationships between different formulas is a very useful research method

Installing the mobile version of MATLAB on a mobile phone can easily verify the correctness of results. MATLAB has efficient computing functions and powerful drawing capabilities, making it easy to calculate and draw with just one phone, simple and convenient. This article hopes to promote the use of the mobile version of MATLAB, allowing more college students to appreciate the beauty of mathematics, thereby increasing their interest in learning mathematics, constantly activating people's creative thinking, and improving their ability to propose and solve problems.

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Disclosure statement

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